ON THE FREQUENCY DOMAIN ANALYSIS OF SWITCHED CAPACITOR NETWORKS INCLUDING ALL PARASITICS

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ABSTRACT

A completely general method to calculate the exact frequency response of switched capacitor (s.c.) filters, including the effects of resistive timeconstants (e.g. due to switch resistances or bandlimited opamps) is presented. These effects are especially important in noise analysis or when designing high Q-filters. The method is based on the wellknown Modified Nodal Analysis (M.N.A.) description and is therefore completely compatible with traditional CAD-methods and with recently developped simulation programs for ideal s.c.-networks.

INTRODUCTION

Many approaches to computer aided analysis of ideal s.c. networks have been proposed [1-6]. None of them is able to handle the influence of bandlimited opamps and finite resistances on the frequency response in a general way. Work in this direction has been reported [7-9] , but is based on state space or is an analytical method and limited to small networks.

However, the parasitic time constants are of prime interest when designing high Q-filters, filters with internal gain or with a small ratio between signal and sampling frequency or when performing noise calculations.

Therefore, a technique, based on the MNA-method, has been developped. The approach is an extension of the method, described in [5-6] , suited for ideal s.c. networks and is completely general. No constraints are placed on the number and the length of the clockphases. Resistive and capacitive networks may be inserted and all frequency domain effects as sin(x)/x and continuous I/O-couplings are included.

The algorithm consists of one z-domain step, coupling the clockphases, and two AC-analyses, which take into account the internal frequency domain behaviour of the networks for each clock phase. The set up of an extended state transition matrix is necessary.

Realistic approximating techniques to handle this are presented. An example will be used to demonstrate the practical interest of the method.

DESCRIPTION OF THE ALGORITHM

Consider a switched capacitor network $\mathcal{N}_s$, controlled by clocksignals, which are T-periodic, ie $\phi_i(t+T)=\phi_i(t)\ \forall\ t,i$ with ϕ_i a Boolean variable, denoting the high and low states of clock i. The time axis can be partitioned into time slots $\Delta_m = (t_m, t_{m+1}]$ such that the clock signals (and hence the network) do not vary in Δ_m. All time slots $\Delta_{k+\ell N}$ for $\ell=0,1,2...$ with the same relative position k in a period (called <u>phase</u> k) have the same duration and the clock values are the same for $t \in \Delta_{k+\ell N}$.

For each phase k, the modified nodal equations can be set up :

$$\begin{vmatrix} \underset{\sim}{G}_k & \underset{\sim}{A}_k \\ \underset{\sim}{B}_k & \underset{\sim}{D}_k \end{vmatrix} \begin{vmatrix} \underset{\sim}{v}(t) \\ \underset{\sim}{i}(t) \end{vmatrix} + \begin{vmatrix} \underset{\sim}{C}_k & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{vmatrix} \begin{vmatrix} \dot{\underset{\sim}{v}}(t) \\ \underset{\sim}{0} \end{vmatrix} = \begin{vmatrix} \underset{\sim}{j}(t) \\ \underset{\sim}{e}(t) \end{vmatrix} \qquad (1)$$

$$\text{or } \underset{\sim}{G}_{ke} \begin{vmatrix} \underset{\sim}{v}(t) \\ \underset{\sim}{i}(t) \end{vmatrix} + \underset{\sim}{C}_{ke} \begin{vmatrix} \dot{\underset{\sim}{v}}(t) \\ \underset{\sim}{0} \end{vmatrix} = \begin{vmatrix} \underset{\sim}{j}(t) \\ \underset{\sim}{e}(t) \end{vmatrix} \qquad (2)$$

$\underset{\sim}{v}(t)$, $\underset{\sim}{i}(t)$, $\underset{\sim}{e}(t)$ and $\underset{\sim}{j}(t)$ respectively stand for the vectors of the nodal voltages, the currents in some selected branches, the voltage and current sources. Note also that in (1), the nodal matrix has been split up in $\underset{\sim}{C}_k$, the nodal capacitance, and $\underset{\sim}{G}_k$, the nodal conductance matrix. For sake of brevity we present the precise statements of the results as theorems and refer the reader for the proofs to the internal report [13].

<u>Lemma 1 : The extended state transition matrix.</u>

For each clockphase k, and for zero input signals, the values of the nodal voltages and selected branch currents at the end of clockphase k can be given in terms of the initial voltages and branch currents as expressed in

$$\begin{vmatrix} \underset{\sim}{v}(t_{k+1}+\ell T) \\ \underset{\sim}{i}(t_{k+1}-\ell T) \end{vmatrix} = \underset{\sim}{P}_k \cdot \begin{vmatrix} \underset{\sim}{v}(t_k+\ell T) \\ \underset{\sim}{i}(t_k+\ell T) \end{vmatrix} \quad \text{for } \ell=0,1,2,.. \qquad (3)$$

with $\underset{\sim}{P}_k = \underset{\sim}{Y}_k(t_{k+1}+\ell T)$ where $\underset{\sim}{Y}_k(t)$ is the <u>extended state transition matrix</u>, which is by definition the solution of differential equation (4) with $\underset{\sim}{C}_{ke}$ and $\underset{\sim}{G}_{ke}$ as defined in (2).

$$\underset{\sim}{C}_{ke} \cdot \dot{\underset{\sim}{Y}}_k(t) + \underset{\sim}{G}_{ke} \cdot \underset{\sim}{Y}_k(t) = \underset{\sim}{0} \qquad (4)$$

with initial condition

$$Y_k(t_k+\ell T) = I \quad (\text{for } \ell = 0,1,2,\dots)$$

Lemma 2 : Final nodal voltages and branch currents for sinusoidal inputs.

For sinusoidal excitations $e(t) = \hat{E}\, e^{j\Omega t}$ and $j(t) = \hat{J}\, e^{j\Omega t}$ the vector of the nodal voltages $v(t_{k+1}+\ell T)$ and some selected branch currents $i(t_{k+1}+\ell T)$ at the end of timeslot $\Delta_{k+\ell N}$ can be expressed as :

$$\begin{bmatrix} v(t_{k+1}+\ell T) \\ i(t_{k+1}+\ell T) \end{bmatrix} = P_k \begin{bmatrix} v(t_k+\ell T) \\ i(t_k+\ell T) \end{bmatrix} + \left[e^{j\Omega t_{k+1}} I - e^{j\Omega t_k} P_k \right] M_k(j\Omega) \begin{bmatrix} \hat{J}_k \\ \hat{E}_k \end{bmatrix} e^{j\Omega \ell T} \qquad (5)$$

with P_k as defined in Lemma 1 and $M_k(j\Omega)$ the inverse of the complex MNA-matrix, evaluated at the input frequency Ω or :

$$M_k(j\Omega) = \left[G_{ke} + j\,\Omega\, C_{ke} \right]^{-1} \qquad (6)$$

Theorem I : the z-transforms $V_k(z)$ and $I_k(z)$ (with $k=1,2,\dots N$) of the sequence of the final nodal voltages and branch currents of a linear T-periodic s.c. network for sinusoidal excitations can be obtained by solving equation (7), where the missing entries are zero and I is the unity matrix.

$$\begin{bmatrix} I & & & & -P_1 z^{-1} \\ -P_2 & I & & & \\ & \ddots & \ddots & & \\ & & -P_{N-1} & I & \\ & & & -P_N & I \end{bmatrix} \begin{bmatrix} X_1^\star \\ X_2^\star \\ \vdots \\ X_{N-1}^\star \\ X_N^\star \end{bmatrix} = S^\star(z).Q(z) \begin{bmatrix} \hat{U}_1 e^{j\Omega t_1} \\ \hat{U}_2 e^{j\Omega t_2} \\ \vdots \\ \hat{U}_{N-1} e^{j\Omega t_{N-1}} \\ \hat{U}_N e^{j\Omega t_N} \end{bmatrix} \qquad (7)$$

with $Q(z)$ equal to

$$\begin{bmatrix} M_1(j\Omega) & & & & -z^{-1} P_1 M_1(j\Omega) \\ -P_2 M_2(j\Omega) & M_2(j\Omega) & & & \cdot \\ \vdots & \cdot & & & \cdot \\ & \cdot & \cdot & & \\ & & -P_{N-1} M_{N-1}(j\Omega) & M_{N-1}(j\Omega) & \\ & & & -P_N M_N(j\Omega) & M_N(j\Omega) \end{bmatrix}$$

with $X_k^\star(z) = \begin{bmatrix} V_k^\star(z) \\ I_k^\star(z) \end{bmatrix}$ and $\hat{U}_k = \begin{bmatrix} \hat{J}_k \\ \hat{E}_k \end{bmatrix}$

and with $S^\star(z) = \sum_{\ell=-\infty}^{+\infty} e^{j\Omega \ell T} z^{-\ell}$

Note that $S^\star(e^{j\omega T}) = \sum_{\ell=-\infty}^{+\infty} e^{j\Omega T \ell}\, e^{-j\omega T \ell} = \omega_s \cdot \sum_{-\infty}^{+\infty} \delta(\omega-\Omega-\ell\omega_s)$ is only non zero for pulsations $\omega = \ell\omega_s + \Omega$ ($\ell = -\infty \dots +\infty$). This is thus also true for $V_k^\star(e^{j\omega T})$ and $I_k^\star(e^{j\omega T})$.

Theorem I is of critical importance in the whole derivation. It allows a computation of the transform of the final node voltages under sinusoidal excitations. It is an extension of the basic theorem of [6] (Theorem II, p. 1031) to non-ideal s.c.-networks. Theorem I requires a previous computation of the P_k-matrices, which are defined for zero excitation. It must be noted that, when using L.U. decomposition to solve (7), techniques described in [10] can be applied, so that only inversion of the lower part of the matrix is necessary for each ω of interest.

To obtain the correct frequency domain behaviour, not only the initial and final values have to be taken in account, but also the intermediate transients (continuous I-O couplings, transient behaviour ...). This is considered in theorem II.

Theorem II : when applying sinusoidal excitations with pulsation Ω to a s.c.-network $\mathcal{N}_s$, the correct frequency domain response for phase k $V_k(j\omega)$, $I_k(j\omega)$ at pulsation $\omega = \Omega + \ell\omega_s$ with $\ell = -\infty,\dots,+\infty$, including the effects of the transients and continuous I/O-couplings, can be found by solving (8) :

$$\underbrace{\left[G_{ke} + j\omega C_{ke}\right]}_{M_k^{-1}(j\omega)} \cdot \begin{bmatrix} V_k(j\omega) \\ I_k(j\omega) \end{bmatrix} = \begin{bmatrix} \hat{J}_k \\ \hat{E}_k \end{bmatrix} . \tau_k(\omega-\Omega) + C_{ke} \left[X_{k-1}(e^{j\omega T}) e^{-j\omega t_k} - X_k(e^{j\omega T}) e^{-j\omega t_{k+1}} \right] \qquad (8)$$

with X_k as obtained from theorem I and

$$\tau_k(\phi) = 2 \sin\left[\phi(t_{k+1}-t_k)/2\right] . \exp\left[j\phi(t_{k+1}-t_k)/2 \,/ T\phi\right] e^{j\phi t_{k+1}} \cdot \sum_{\ell=-\infty}^{+\infty} (\phi - \ell\omega_s) \qquad (9)$$

with $\omega_s = 2\pi/T$.

Observe that $V_k(j\omega)$, $I_k(j\omega)$ only has a nonzero contribution at pulsations $\omega = \Omega + \ell\omega_s$ with $\ell = -\infty \dots +\infty$

The total frequency response, is obtained by summing over the N phases

$$V(j\omega) = \sum_{k=1}^{N} V_k(j\omega) \qquad (10)$$

Summary :

The setup of an algorithm, calculating the frequency response of nonideal s.c.-networks is now straightforward :

- calculation of $M_k(j\Omega)$ at the input frequency
- evaluation of the z-domain series of the final node voltages and branch currents of each phase, using (7)
- the resulting series serve as input to the ac-analyses of (8), giving the final frequency response at

EVALUATION OF THE EXTENDED STATE TRANSITION MATRIX.

Equation (3) clearly accentuates the practical importance of the P_k-matrix. It describes how the final value in an interval can be computed from the initial values under zero-excitations. P_k is a constant matrix, whose value depends on C_{ke}, G_{ke} and $\Delta_k (=t_{k+1}-t_k)$, as can be derived from (4). Note however that G_{ke} and/or C_{ke} may be singular. Due to these singularities, many classical methods for calculating matrix exponentials, cannot be used. Two approximating methods, depending upon the desired results, will be presented.

General Solution (non-equilibrium).

Matrix exponential-approximating techniques, which can handle singular matrices, can be used to solve the differential equation (4). Among the many existing techniques [12], the Padé approximation seems to be best suited for this application : e.g. for first order Padé apprixmation (or trapezoidal)

$$P_k \cong \left[\left\{ C_{ke} + \frac{h}{2} G_{ke} \right\}^{-1} \cdot \left\{ C_{ke} - \frac{h}{2} G_{ke} \right\} \right]^m \quad \text{with } h = \frac{\Delta_k}{m} = \frac{t_{k+1} - t_k}{m} \qquad (11)$$

This expression is always computable, due to the fact that $C_{ke} + \frac{h}{2} G_{ke}$ is non-singular. When choosing m high enough to ensure accuracy for the dominant time-constants, first or higher order Padé-approximations lead to a stable and accurate algorithm to approximate P_k.

Computation of P_k using the equilibrium principle.

In most of the commonly used sc-systems, it may be assumed that the transient effects, which occur after each switching instant, disappear completely before the end of the particular timeslot and that the different node voltages reach their equilibrium (under zero input conditions). The basic philosophy is to invoke this equilibrium in order to assume that at the end of the timeslot the capacitor voltages do not vary. So they can be replaced by open circuits. The remaining equations, which describe the resistive networks in equilibrium, are not sufficient. They have to be completed with a set of capacitor charge equations, leading to a complete set of equations, describing the equilibrium state. These capacitor equations can be set up using a topological approach, which uses the same ideas as these of [11].

Consider a s.c.-network $\mathcal{N}_s$, which may contain ideal and non-ideal switches, capacitors, resistors, current and voltage sources and all controlled sources. No cutsets of capacitors and current sources are allowed. A two step algorithm is presented for the computation of P_k of phase k.

STEP 1 : Consider all cutsets of $\mathcal{N}_{sk}$ ($\mathcal{N}_s$ for phase k), containing only capacitors and open switches. These cutsets divide $\mathcal{N}_{sk}$ in ℓ different parts, which are called $\mathcal{N}_{sk}^i$ and which contain n_i nodes, $(i=1,\ldots \ell)$.

A complete independent set of such cutsets can be found by inspecting the non reduced incidence matrix, and rearranging rows and columns. Rearranging the MNA equations in the same way, grouping the nodes of $\mathcal{N}_{sk}^i$ $(i=1,\ldots\ )$ a new set of equations is obtained :

$$\begin{bmatrix} C'_k & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{v}' \\ \dot{i} \end{bmatrix} + \begin{bmatrix} G'_k & A'_k \\ B_k & D_k \end{bmatrix} \cdot \begin{bmatrix} v' \\ i \end{bmatrix} = \begin{bmatrix} j' \\ e \end{bmatrix} \qquad (12)$$

STEP 2 : Removal of the dependent equations when the equilibrium principle is valid.

Consider the Kirchoff current equations of (13) which are related to the n_i nodes inside the subnetwork $\mathcal{N}_{sk}^i$

$$C'_{ki}\ \dot{v}'(t) + G'_{ki}\ v'(t) + A'_{ki}\ i(t) = j'_i(t) \qquad (13)$$

Choose a reference node for each $\mathcal{N}_{sk}^i$ (suppose we choose the last node), except for the subnetwork containing the ground reference node. Dropping the Kirchoff current equation of the reference node of $\mathcal{N}_{sk}^i$ and replacing it by the sum of all Kirchoff current equations of $\mathcal{N}_{sk}^i$ (Note that this equation is purely capacitive, by the choice of the cutsets), (13) is equivalent with (14-15) :

$$n_i - 1 \left\{ C''_{ki}\, \dot{v}(t) + G''_{ki}\, v'(t) + A''_{ki}\, i(t) = j''_i(t) \right. \qquad (14)$$

$$1 \left\{ C'''_{ki}\, \dot{v}'(t) = 0 \right. \qquad (15)$$

with $C'''_{ki}(1,m) = \sum_{\ell=1}^{n_i} C'_{ki}(\ell,m)$ for $m=1,\ldots n_i$

and $C''_{ki}(\ell,m) = C'_{ki}(\ell,m)$ for $m=1,\ldots n_i$ and $\ell=1,\ldots n_i-1$

Integrating equation (15), which is precisely the cutset equation and supposing that the equilibrium principle is valid, which implies that all node voltages at the end of the timeslot are constant for zero inputs or $\dot{v}\,(t_{k+1}+\ell T)=0$, equations (14) and (15) are transformed into (16) and (17)

$$G''_{ki}\, v'(t_{k+1}+\ell T) + A''_k\, i(t_{k+1}+\ell T) = 0 \qquad (16)$$

$$C'''_k\, v'(t_{k+1}+\ell T) = C'''_k\, v'(t_k+\ell T) \qquad (17)$$

By the choice of the cutsets and the setup of the equations, it is clear that (16) and (17) give an independent set of equations. Repeating this for all $\mathcal{N}_{sk}^i$, combining equations (16), (17) and the voltage equations of (12), leads to (18)

$$\begin{vmatrix} G''_k & A''_k \\ C'''_k & 0 \\ B'_k & D'_k \end{vmatrix} \begin{vmatrix} v'(t_{k+1}+\ell T) \\ i\ (t_{k+1}+\ell T) \end{vmatrix} = \begin{vmatrix} 0 \\ C'''_k \\ 0 \end{vmatrix}\ v'(t_k\ +\ell T) \qquad (18)$$

or

$$P_k = \begin{vmatrix} G''_k & A''_k \\ C'''_k & 0 \\ B'_k & D'_k \end{vmatrix}^{-1} \cdot \begin{vmatrix} 0 \\ C'''_k \\ 0 \end{vmatrix} \qquad (19)$$

Following these steps, an algorithm can be set up easily. It must be emphasized that this equilibrium principle is valid for almost all practical sc-circuits and is interesting for the calculation of noise transferfunctions, high Q filters,...

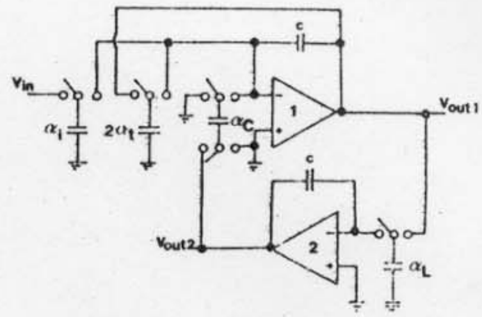
EXAMPLE

The frequency response of the high Q (Q=100)-biquad bandpass filter of Fig. 1 is shown in Fig.2. (f_s=200Khz, f_o=12.7Khz). Fig. 2 gives the frequency domain response of node V_{out2}. Curves a&b respectively show the responses with ideal opamps and opamps with limited gain (10K), while curve c is the response with non-ideal opamps (f_p=100Hz, A=10K). The large deviation of V_{out2} (c) is caused by the huge loading of opamp2 (α_L=78.5). This simulation was realised with a special test program, not using sparse matrix techniques.

CONCLUSIONS

A completely general method for the analysis of switched capacitor systems in the frequency domain has been presented. The effects of all nonidealities as limited opamp-bandwidths, resistances, continuous I/O couplings, resistive dividers and time constants, ... can be studied.

The presented method is completely compatible with existing CAD-methods for ideal sc networks and ac analysis and is particularly suited for noise analysis of sc systems.



f_s : 200Khz
opamp-gain : 10K
Unity-gain B.W. : 1 Mhz
α_L : 78.5
$\alpha_i\ \alpha_t\ \alpha_C$: 0.00196

FIG. 1

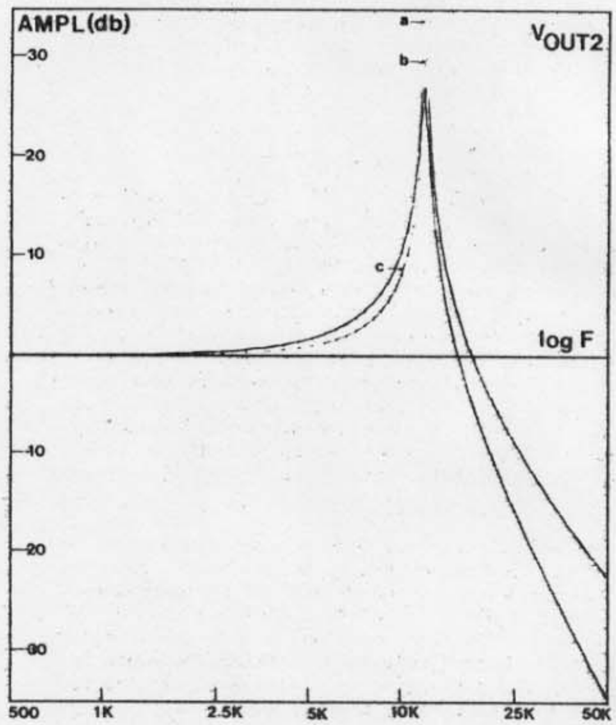


Fig 2

REFERENCES

[1] F. Brglez :"Exact nodal analysis of switched-capacitor circuits with arbitrary switching sequences and general inputs". Part I - IEEE Proc. of 12th Asilomar Conf. on Circ. Syst. and Comp., 6-8 Nov. 1978. Part II - Proc. of ISCAS '79, July 17-19, pp. 748-751.

[2] C.F. Kurth, G.S. Moschytz :"Two-port analysis of switched-capacitor networks using four-port equivalent circuits in the z-domain". IEEE Trans. Circuits Syst., Vol.CAS-26, pp. 166-180, March 1979.

[3] M.L. Liou, Y.K.Kuo :"Exact analysis of switched-capacitor circuits with arbitrary inputs". IEEE Trans. Circuits Syst., Vol. CAS-26, pp.213-223, April 1979.

[4] Y.P. Tsividis :"Analysis of switched capacitive networks". IEEE Trans. Circuits Syst., Vol. CAS-26, pp. 935-946, Nov. 1979.

[5] H. De Man, J. Rabaey, G. Arnout and J. Vandewalle: Practical implementation of a general computer aided design technique for switched capacitor circuits". IEEE J. of Solid State Circuits, Vol. SC-15, April 1980, pp. 190-200.

[6] J. Vandewalle, H. De Man, J. Rabaey :"The adjoint switched capacitor network and its applications", Proc. of ISCAS 80, Houston, pp. 1031-1034.

[7] T. Ström and S. Signell :"Analysis of periodically switched linear circuits". IEEE Trans. Circuits Systems, Vol. CAS-24, pp. 531-541.

[8] H. Gutsche :"On the analysis of switched capacitor circuits including parasitic elements", Proc. of ISCAS 80, Houston, pp. 973-976.

[9] D. van Grünigen, U.W. Brugger :"Closed-form analysis of nonideal switched-capacitor networks", Proceedings of ECCTED 80, Warsaw, pp.165-170.

[10] J. Vandewalle, L. Claes en, H. De Man :"Very efficient computer algorithms for direct frequency, aliasing and sensitivity analysis of switched capacitor networks". ISCAS 81, Chicago.

[11] S.C. Fang, Y.P. Tsividis :"Modified nodal analysis with improved numerical methods for switched capacitive networks". Proc. of ISCAS 80, Houston, pp. 977-980.

[12] C. Moler and C. Van Loan :"Nineteen dubious ways to compute the exponential of a matrix", SIAM Review, Vol. 20, no. 4, Oct. 1978, pp. 801-836.

[13] Report ESAT Laboratory, Katholieke Universiteit Leuven, Belgium, 1981.

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