An Equation-based Method for Phase Noise Analysis

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Abstract

A method for calculating phase noise based on a particle diffusion model in the phase plane is introduced. This approach allows for the derivation of closed-form solutions, which can be used to see the explicit dependencies of phase noise on circuit component values. The theory is applied to the Van der Pol and Colpitts oscillators, and comparisons to simulations show close matching.

Introduction

The need for a regular time base has made the oscillator an important component of many systems. Unfortunately, because of the presence of noise, no real oscillator can be truly periodic. Instead, the actual period of oscillation fluctuates about some mean period. This is manifested in frequency space as a peak about the mean frequency with spectral content in nearby frequencies, instead of a delta function as in a truly periodic oscillator. This spread is referred to as phase noise. Many have looked at calculating this spread [5] [6] [7] [8] [9]. Most of these methods rely on linear techniques, which fail to capture the inherently nonlinear effects of oscillators. One fully nonlinear model [9] is effective in calculating phase noise but focuses on exact solutions, making it complicated and thus more suitable for simulators. Recently, a similar view to that presented in this paper was published [5] explaining the phase diffusion with fluctuation-dissipation methods. While the concepts are similar, an equation-based method to handling time-varying noise was not made explicit. This paper presents a perturbative method for calculating this spread while accounting for nonlinearities in the oscillator, which is based solely on hand analysis and equations, and results in approximate closed-form expressions.

Phase Plane Perspective

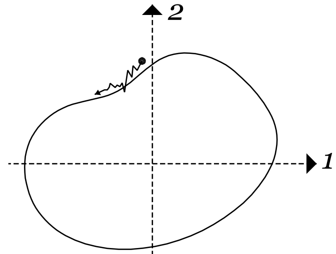

Fig. 1. Path of a noisy oscillator in phase space.

A point in phase space characterizes the complete state of the system, allowing for predictions of the past and future of the deterministic system. As time passes, the state of an oscillator traces out a trajectory which tends towards an isolated closed orbit called a limit cycle. A single noisy oscillator would consist of a "jagged" (see Fig. 1) orbit as noise disturbs the otherwise smooth path of the oscillator's state. Solving for the actual path by considering the fluctuations of the noise and its effects on the trajectory can get very involved. One can also solve the same problem by considering the flow of the ensemble of oscillators (see Fig. 2). Solving for the density of realizable oscillators instead of the one realized path transforms the problem into a more tractable problem. The resulting equation describing the density flow is none other than the advection-diffusion equation [4] with non-constant coefficients, which are due to the non-constant flow field (restoring forces and damping) in the phase plane. This is due to the nonlinear behavior of the oscillator, which presents damping that is a function of the actual voltage in the tank. This PDE is sometimes referred to as the Fokker-Planck equation.

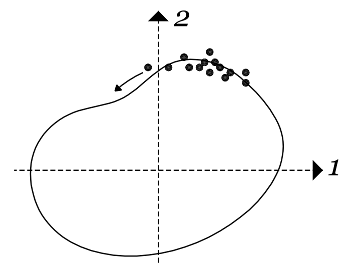

Fig. 2. Ensemble of oscillators flowing through phase space around the limit cycle.

Perturbation Analysis

The main idea behind perturbation theory [1] [2] [3] [4] is solving a problem by expanding about nearby known solutions. For our problem, two limiting cases become obvious for further inspection: oscillations with strong and weak nonlinearities. This paper focuses on the latter case, which exhibits near sinusoidal solutions.

In order to analyze a variety of oscillator circuit topologies, we solve for a general nonlinearity, f, which is also assumed to be time-independent. We find the autocorrelation function of the output voltage which can then be transformed into the power spectral density for the phase noise. Starting with the non-dimensionalized differential equation,

$$\ddot{q}+q=\epsilon f\left(q,\dot{q}\right)+w(t) \tag{1}$$

where ϵ is small and $w(t)$ is a white noise source, which we assume to be small compared to the amplitude of oscillation. The natural frequency of this oscillator is 1 because of the non-dimensionalization. Now turning this into a system of first-order equations, we have

$$\begin{aligned}\dot{q}&=v\\ \dot{v}&=-q+\epsilon f\left(q,v\right)+w(t)\end{aligned} \tag{2}$$

The two phase space variables are q and v (which are proportional to the charge and voltage in parallel-tank electrical oscillators). Their dynamics characterize the oscillator. The associated advection-diffusion equation [4] describing the flow of the phase space density is then

$$\eta_t+\nabla\cdot\left(\left(v\mathbf{1}-q\mathbf{2}\right)\eta+\epsilon f\left(q,v\right)\eta\mathbf{2}-\epsilon D\eta_v\mathbf{2}\right)=0 \tag{3}$$

where η is the phase space probability density, D is the diffusion constant which depends on the magnitude of the noise, and $\mathbf{1}$ and $\mathbf{2}$ are the unit vectors in the q- and v-directions, respectively. We pulled out a factor of ϵ from the diffusion term because the noise is assumed to be $O(\epsilon)$. To solve this, we consider solutions of the form

$$\eta(t,q,v)=\eta^0(t,q,v)+\epsilon\eta^1(t,q,v)+\cdots \tag{4}$$

where the superscripts are not to be confused with exponents. Equating coefficients, we get

$$\eta^0_t+\nabla\cdot\left(v\mathbf{1}-q\mathbf{2}\right)\eta^0=0 \tag{5}$$

which is just the equation for a linear harmonic oscillator with sinusoidal solutions. Using the method of multiple scales [1] [2] [3] [4], we introduce a slow time variable $T\equiv\epsilon t$. We also convert to polar coordinates to get the first-order equation

$$\eta^1_t-\eta^1_\theta+\eta^0_T=-\nabla\cdot\left\{\left(f\left(q,v\right)\eta^0-D\eta^0_v\right)\mathbf{2}\right\} \tag{6}$$

Changing to a rotating frame of reference by defining a new variable $\phi=\theta+t$ removes the θ-dependence in η^0 since the unperturbed linear oscillator rotates at exactly frequency 1, and thus η^0 is only a function of $\theta+t$. Hence,

$$\eta^1_\theta-\eta^0_T=\nabla\cdot\{A\mathbf{2}\}+\frac{1}{r}\partial_\phi\left\{A\mathbf{2}\cdot\hat{\theta}\right\} \tag{7}$$

where

$$A=\left(f\left(r,\theta\right)\eta^0-D\sin\theta\eta^0_r-\frac{D}{r}\cos\theta\eta^0_\phi\right) \tag{8}$$

and the nonlinearity f is written in terms of polar coordinates, r and θ. Now integrating (7) over the period $[0,2\pi)$, we get

$$\eta^0_T=-\frac{1}{2\pi r}\partial_r\int A\mathbf{2}\cdot\mathbf{r}\;r\,d\theta-\frac{1}{2\pi r}\partial_\phi\int A\mathbf{2}\cdot\hat{\theta}\,d\theta \tag{9}$$

where the η^1_θ term integrates to zero because η is 2π-periodic in θ. To find the phase noise, we need to find the conditional probability that a phase dot will move from one point in space to another. This will enable the calculation of the autocorrelation function, which can then be transformed to the power spectral density. Hence, we need to find the transient response of the phase diffusion given an initial condition. Assuming the oscillation frequency is close to 1, the phase drift term will be small compared to the phase diffusion term in this rotating frame. Substituting (8) into (9) and neglecting the amplitude drift and diffusion terms, we find the phase diffusion PDE,

$$\eta^0_T\sim\frac{\eta^0_{\phi\phi}}{2\pi r_c^2}\int_0^{2\pi}D\cos^2\theta\;d\theta \tag{10}$$

with initial condition

$$\eta^0(\phi,0)=\delta(\phi),\quad \phi\in[-\pi,\pi) \tag{11}$$

where r_c is the limiting amplitude of oscillation. We also made use of the fact that η^0 is independent of θ to pull it out of the integral. This is just the heat equation with solution

$$\eta^0(\theta,t)=\frac{1}{2\pi}\sum_k e^{-\frac{D'}{r_c^2}k^2\epsilon t}e^{ik\theta}e^{ikt} \tag{12}$$

where we have replaced the slow time variable, T, with ϵt, and converted back into the non-rotating frame (using $\phi=\theta+t$). We also defined

$$D'=\frac{1}{2\pi}\int_0^{2\pi}D\cos^2\theta\;d\theta \tag{13}$$

We compute the autocorrelation function of v,

$$\begin{aligned}R_v(t)&=r_c^2\left\langle\sin(\alpha)\sin(\alpha+\theta)\right\rangle\\ &=r_c^2\left\langle\sin^2(\alpha)\cos(\theta)+\sin(\alpha)\cos(\alpha)\sin(\theta)\right\rangle\end{aligned} \tag{14}$$

where α is the phase at some time t' and θ is the phase difference at a time $t'+t$ later. Over long times, (when the oscillator reaches steady-state) α will be uniformly distributed over all angles. The process is also assumed to be wide-sense stationary, making $R_v(t)$ a function of the time difference only. Consider

$$\Re\left\langle e^{i\theta}\right\rangle=\Re\left[e^{-\frac{D'}{r_c^2}\epsilon t}e^{-it}\right]=e^{-\frac{D'}{r_c^2}\epsilon t}\cos t \tag{15}$$

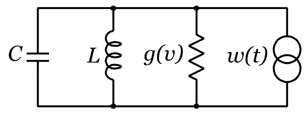

Fig. 3. Van der Pol oscillator with white noise source.

where only the $k = -1$ term survives. Thus,

$$R_v(t) = \frac{r_c^2}{2} e^{-\frac{D'}{r_c^2}\epsilon t} \cos t \tag{16}$$

Taking the Fourier transform of (16) results in the power spectral density of the phase noise.

Van der Pol Oscillator

Now we'll consider the weakly nonlinear Van der Pol oscillator as an example (shown in Fig. 3). Consider the oscillator with a negative resistance device with *I*-*V* characteristic

$$i = -\frac{1}{R}\left(1 + \frac{1}{3}a_1 v^2\right) v \tag{17}$$

The associated differential equation would then be

$$-C\dot{v} - \int \frac{v}{L}\, dt = -\frac{1}{R}\left(1 + \frac{1}{3}a_1 v^2\right) v + w(t) \tag{18}$$

Defining $q = \int v\, dt$ and non-dimensionalizing with

$$\frac{t}{\sqrt{LC}} \to t \qquad \frac{q}{\sqrt{\frac{3}{-a_1}}\sqrt{LC}} \to q \tag{19}$$

gives

$$\ddot{q} + q = \frac{L/R}{\sqrt{LC}} \left(1 - \dot{q}^2\right) \dot{q} + \bar{w}(t) \tag{20}$$

where

$$\bar{w}(t) = -\frac{w(t)}{\sqrt{\frac{C}{L}}\sqrt{\frac{3}{-a_1}}} \tag{21}$$

We see the small parameter comes out to be $\epsilon = \frac{L/R}{\sqrt{LC}} = 1/Q$. Thus, high-$Q$ circuits which tend to have nearly sinusoidal oscillations are weakly nonlinear, which is what we expected. From (20), the nonlinearity is

$$f(q, v) = \left(1 - v^2\right) v \tag{22}$$

From (16), and putting the units back in with (19), the power spectral density can be found to be

$$\mathrm{F}\left[\frac{r_c^2}{2} e^{-\frac{1}{2}\frac{D}{r_c^2}\epsilon\frac{t}{\sqrt{LC}}} \cos\frac{t}{\sqrt{LC}}\right] \approx \frac{\frac{1}{4}D\epsilon\frac{1}{\sqrt{LC}}}{\Delta\omega^2} \tag{23}$$

where constant noise variance was assumed (D is constant). The power spectral density is Lorentzian[1] and behaves like the expected $1/\Delta\omega^2$ far from the carrier. To get the final expression for phase noise, D must be calculated in terms of the circuit component values. Based on the derivation of the diffusion equation [4],

$$\epsilon D = \frac{1}{2}\left\langle \bar{w}(t)^2 \right\rangle = \frac{1}{2}\left\langle w(t)^2 \right\rangle \frac{L}{C}\left(\frac{-a_1}{3}\right) \tag{24}$$

However, this is still in normalized units. Denormalizing, we get

$$\epsilon D = \frac{kT}{R}\sqrt{\frac{L}{C}}\frac{1}{C} \tag{25}$$

where we used $2kT/R$ for $\langle w(t)^2 \rangle$ and remembering that ϵD has the units of v^2/t since the diffusion is in the v-direction. Thus, the single-sided spectral density is

$$S_v(\Delta\omega) \approx \frac{\frac{1}{2}\frac{kT}{RC^2}}{\Delta\omega^2} \tag{26}$$

Interpreting (26), it is clear that increasing R or C reduces phase noise. Larger resistance implies smaller current noise which translates to lower noise overall. Larger capacitance implies a smaller diffusion constant resulting in a more sluggish or inert oscillator (when viewed as the flow of the ensemble), which translates to a smaller susceptibility to being pushed around by noise disturbances. Note that the increase in C affects the overall phase noise more than the increase in R. This is of particular interest when considering the problem of finite-Q inductors. Assuming that in a given process, inductors can only achieve a certain maximum Q, the inductance, L, will then be proportional to R (since the frequency of oscillation is fixed)[2]. Hence, from (26), increasing C and minimizing L results in lower phase noise. There is, however, a limit to how small the inductance can be made to be in practical oscillators, which will be determined by startup conditions.

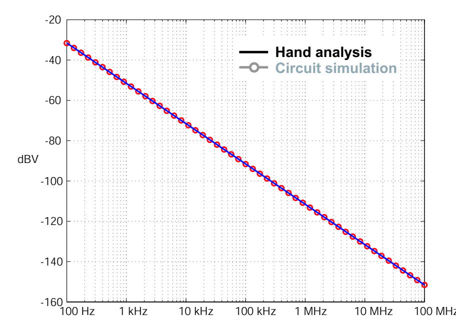

Fig. 4. Phase noise plots of voltage-independent noise for a Van der Pol oscillator.

[1] More precisely, there are absolute value signs around t since forward and reverse time are equivalent.

[2] $Q = \omega L/R$ for parallel tanks.

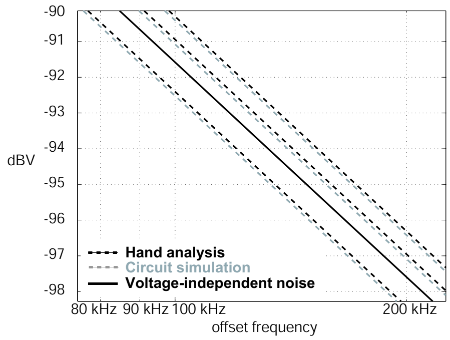

Fig. 5. Phase noise plots of voltage-dependent noise for three different modified Van der Pol oscillators.

The comparison between hand analysis and SpectreRF (`pnoise`) simulations for phase noise is shown in Fig. 4 for the case of constant (independent of tank voltage) noise. The solid line marks hand-calculated values, whereas the circles represent simulated values. The difference between the two curves is about 0.005 dB. When the noise variance is made dependent on the actual resonator voltage, we see that the phase noise changes as shown in Fig. 5. The hand-analytical solutions differ from computer simulations (dark dotted lines are hand-calculated, light dotted lines are simulated) by about 0.2 dB. For verification, we chose to simulate with three modified Van der Pol oscillators with nonlinearities

$$f(q, v) = (1 - v^n)\, v \tag{27}$$

where $n = 2, 4, 6$. As a reference, the constant noise case is shown as a solid black line. Also note that the frequency axis has been scaled (zoomed in) to show the difference between the three curves.

Colpitts Oscillator

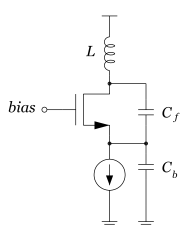

Fig. 6. CMOS Colpitts oscillator.

As an example of using our formalism on a less idealized circuit, consider the CMOS Colpitts oscillator. The nonlinearities in a Colpitts oscillator are more complicated than the simple Van der Pol because of the complexity of the MOS transistor. For analysis, the EKV transistor model [10] [11] is used to take into account all regions of inversion of the MOS transistor. Fig. 7 shows a comparison between hand analysis and Spectre simulations of a Colpitts oscillator. The Spectre simulations used ST's 0.13 μm BSIM 3v3 models, which were then fit to an EKV model for use in hand analysis. The equations underestimate output noise by about 2 dB and ignore time-varying

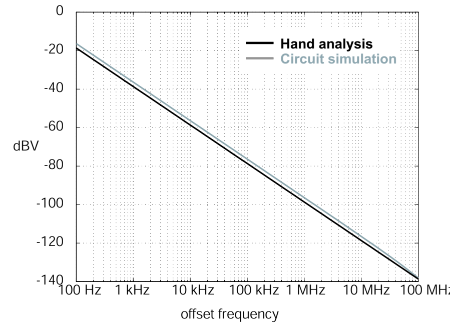

Fig. 7. Phase noise in a CMOS Colpitts oscillator with voltage-dependent noise neglected.

noise effects and is given by,

$$S_v(\Delta\omega) \approx \frac{\frac{1}{2}\frac{kT\gamma g_m}{C^2}}{\Delta\omega^2}$$

Conclusion

An equation-based method for calculating phase noise which results in simple closed-form solutions was presented. Perturbation theory was used to get analytical solutions without the need for simulators or fudge factors. Solutions were compared with circuit simulations for a few modified Van der Pol oscillators and the Colpitts oscillator.

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References

[1] J. D. Cole, *Perturbation Methods in Applied Mathematics*. Blaisdell Publishing Company, 1968.

[2] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*. Springer, 1999.

[3] R. H. Rand, *Lecture Notes on Nonlinear Vibrations*. online book at http://www.tam.cornell.edu/randdocs/nlvib36a.pdf, 2001.

[4] J. C. Neu, *Ma 204 Lecture Notes*. personal handouts, 1999.

[5] D. Ham and A. Hajimiri, "Virtual damping and Einstein relation in oscillators," *IEEE Journal of Solid-State Circuits*, vol. 38, pp. 407–418, March 2003.

[6] D. B. Leeson, "A simple model of feedback oscillator noise spectrum," *Proc. IEEE*, vol. 54, pp. 329–330, Feb. 1966.

[7] B. Razavi, "A study of phase noise in CMOS oscillators," *IEEE. J. Solid-State Circuits*, vol. 31, pp. 331–343, Mar. 1996.

[8] A. Hajimiri and T. H. Lee, "A general theory of phase noise in electrical oscillators," *IEEE J. Solid-State Circuits*, vol. 33, pp. 179–194, Feb. 1998.

[9] A. Demir, A. Mehrotra, and J. Roychowdhury, "Phase noise in oscillators: A unifying theory and numerical methods for characterization," *IEEE Trans. Circuits Systems I*, vol. 47, pp. 655–674, May 2000.

[10] C. C. Enz, F. Krummenacher, and E. A. Vittoz, "An analytical MOS transistor model valid in all regions of operation and dedicated to low-voltage and low-current applications," *Analog Integrated Circuits and Signal Processing*, vol. 8, pp. 83–114, July 1995.

[11] Y. Tsividis, *Operation and Modeling of the MOS Transistor*. McGraw-Hill, 1999.