HOMEWORK 10
SOLUTION

a) At low frequencies C_L sees R_{out} and the C_p at the source of M_8 sees a resistance on the order of r_o.

Since $R_{out} >> r_o$ (and since it is reasonable to assume $C_L >> C_p$), the dominant pole will be $\boxed{p_d = \frac{1}{C_L R_{out}}}$

At frequencies well beyond the dominant pole frequency, the resistance at the source of M_8 and M_7 looks like $\frac{1}{g_{m7,8}}$

(the output will be shorted by C_L)

$\rightarrow \boxed{p_{nd} = \frac{g_{m7,8}}{C_p}}$

b) The total phase as a function of frequency, well beyond the dominant pole, is given by:

$$\underbrace{-90^\circ}_{\text{dominant pole}} - \operatorname{atan}\frac{\omega}{p_{nd}}$$

For $\omega = GBW = \omega_d \cdot A_o \cdot f = \frac{1}{C_L R_{out}} g_{m1,2} R_{out} \cdot f$

$$= \frac{f \cdot g_{m1,2}}{C_L}$$

This phase is also equal to

$$-180^\circ + PM$$

(where PM is the phase margin)

(strictly speaking, the GBW is only the approximate frequency at which the (loop) gain becomes unity, but we will ignore the difference)

$$\Rightarrow \tan PM = \frac{\omega_{nd}}{GBW}$$

So for $PM \geq 60^\circ$, $\boxed{\frac{\omega_{nd}}{GBW} \geq \sqrt{3} = 1.73}$

$$\Rightarrow \quad \boxed{\frac{1}{f}\,\frac{g_{m7,8}}{g_{m1,2}}\,\frac{C_L}{C_p} \geq \sqrt{3} = 1.73}$$

c) $$A(s) = \frac{A_o}{\left(1+\frac{s}{p_d}\right)\left(1+\frac{s}{p_{nd}}\right)}$$

closed loop gain $$a(s) = \frac{A(s)}{1+f\cdot A(s)}$$

$$\Rightarrow a(s) = \frac{A_o}{1+fA_o}\;\frac{1}{1+s\left(\frac{1}{(1+fA_o)p_d}+\frac{1}{(1+fA_o)p_{nd}}\right)+\frac{s^2}{(1+fA_o)p_d p_{nd}}}$$

$$\boxed{a(s) \approx \frac{A_o}{1+A_o f}\;\frac{1}{1+\frac{s}{GBW}+\frac{s^2}{GBW\cdot p_{nd}}}}$$

for $PM = 60°$: $p_{nd} = \sqrt{3}\,GBW$

and $$a(s) \approx \frac{A_o}{1+A_o f}\;\frac{1}{1+\frac{s}{GBW}+\frac{s^2}{\sqrt{3}\,GBW}}$$

d) we can reformulate the closed loop gain as

$$a(s) = \frac{a_0}{1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}}$$

where $a_0 = \dfrac{A_0}{1 + f A_0} = \dfrac{1}{f}\left(1 - \dfrac{1}{1 + A_0 f}\right)$

$$\omega_n = \sqrt[4]{3}\, GBW$$

$$\zeta = \frac{1}{2} \sqrt[4]{3} < 1 \longrightarrow \text{underdamped system}$$

$\Rightarrow$ step response $\left(= \mathcal{L}^{-1}\left\{ \frac{a(s)}{s} \right\}\right)$

$$= \frac{1}{f}\left(1 - \frac{1}{1 + A_0 f}\right)\left(1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \sin\left(\sqrt{1 - \zeta^2}\, \omega_n t + \varphi\right)\right).$$

$$= \frac{1}{f}\left(1 - \varepsilon_{static}\right)\left(1 - \varepsilon_{dynamic}\right)$$

ideal closed loop gain

static error, due to finite loopgain at low frequencies

dynamic error, due to incomplete linear settling

$$\varepsilon_d = \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \varphi\right)$$

$$|\varepsilon_d| \leq \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} = \frac{e^{-t/\tau}}{\sqrt{1-\zeta^2}}$$

τ = time constant of the envelope of the settling error

$$= \frac{1}{\zeta \omega_n} = \frac{2}{\sqrt{3}\ GBW} \gtrsim \frac{1}{GBW}$$